Effective theory for $\omega \ll k \ll gT$ color dynamics in hot non-Abelian plasmas

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A proper sequence of effective theories, corresponding to larger and larger distance scales, is crucial for analyzing real-time equilibrium physics in hot non-Abelian plasmas. For the study of color dynamics (by which I mean physics involving long wavelength gauge fluctuations), an important stepping stone in the sequence of effective theories is to have a good effective theory for dynamics with wave number k well below the Debye screening mass. I review how such dynamics is associated with inverse time scales $\omega \ll k$. I then give a compact way to package, in the $\omega \ll k$ limit, Bödeker's description of $k \ll m$ physics, which was in terms of Vlasov equations with collision terms. Finally, I show how the resulting effective theory can be reformulated as a path integral.

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I. INTRODUCTION

The color fluctuations of very hot, weakly coupled, non-Abelian plasmas are non-perturbatively large at distance scales R of order $(g^2T)^{-1}$. Their dynamics is of particular interest because it is responsible for the large rate of baryon number violation in hot electroweak theory, and so lies at the heart of electroweak scenarios for baryogenesis. ''Hot'' here means hot enough to (a) be ultra-relativistic, (b) ignore chemical potentials, and (c) be in the hot, symmetric phase if there is a Higgs mechanism. It is now known [1] that the time scale associated with non-perturbative color dynamics is $t \sim [g^4T\ln(1/g)]^{-1}$, which is long in the sense that $t \gg R$ (in the weakly coupled limit). Equivalently, the spatial momentum and the frequency scales associated with non-perturbative color dynamics are

$$k \sim g^2T, \quad \omega \sim g^4T\ln(1/g). \tag{1.1}$$

This momentum scale k is small compared to the Debye mass

$$m \sim gT. \tag{1.2}$$

The goal of this paper is to present an effective theory for color dynamics on scales $\omega \ll k \ll m$, to formulate that effective theory solely in terms of gauge fields $A_\mu(t,\mathbf{x})$, and to write the effective theory in path integral form.

It has been known for some time [2] how to write a leading-order effective theory for color dynamics at the scale $k \sim m \sim gT$, where leading-order means that corrections are suppressed by powers of g. The zero-temperature non-Abelian Maxwell equations are modified by what are known as ''hard thermal loops,'' which incorporate the effects of interactions of the soft $k \sim gT$ degrees of freedom with hard $k \sim T$ thermal excitations in the plasma. There is a standard way of writing this effective theory which has a simple physical interpretation [3]. One treats the soft fields classically, and replaces the hard excitations by classical distribution functions $n(\mathbf{x},\mathbf{p},t)$ which describe the density of hard excitations at position $\mathbf{x}$ with momentum $\mathbf{p}$. Writing down Maxwell's equations, together with an appropriately gauge-covariant, linearized Boltzmann equation for n, then produces the leading-order effective theory. n is a density matrix in color space, and the piece of it that's relevant to long-distance color dynamics (at leading order) is the adjoint color piece. It is also convenient and conventional to integrate this adjoint piece over the magnitude $|\mathbf{p}|$ of momentum, replacing $n(\mathbf{x},\mathbf{p},t)$ by an adjoint field $W(\mathbf{x},\mathbf{v},t)$, where $\mathbf{v} \equiv \hat{\mathbf{p}}$. The resulting equations, if W is given a convenient overall normalization, are [4]

$$(D_t + \mathbf{v}\cdot\mathbf{D})W - \mathbf{v}\cdot\mathbf{E} = 0, \tag{1.3a}$$

$$D_\nu F^{\mu\nu} = j^\mu = m^2 \langle v^\mu W \rangle_{\mathbf{v}}, \tag{1.3b}$$

where $m \sim gT$ is again the leading-order Debye mass, $\langle \cdots \rangle_{\mathbf{v}}$ denotes angular averaging over the direction $\mathbf{v}$, and $v^\mu \equiv (1,\mathbf{v})$. Formally solving the Boltzmann equation for W and plugging the result into the Maxwell equation, one obtains the hard-thermal loop equation of motion for the soft gauge field, which is

$$D_\nu F^{\mu\nu} = j^\mu = m^2 \langle v^\mu (D_t + \mathbf{v}\cdot\mathbf{D})^{-1} \mathbf{v}\cdot\mathbf{E} \rangle_{\mathbf{v}}. \tag{1.4}$$

This equation contains, among other things, the physics of Debye screening, which screens static electric fields over distances of order $1/m$.

A qualitatively important point [5] can be extracted from Eq. (1.4): $k \ll m$ physics is dominated by frequencies $\omega \ll k$. For the sake of quickly reviewing this point here, focus for simplicity on the linear terms on the right-hand side of Eq. (1.4), focus on their $\omega \ll k$ behavior, and let us check self-consistently that the dominant frequency falls in the $\omega \ll k$ regime. Focus in particular on the transverse modes of the gauge field, which are not Debye screened for $k \ll m$. In the $\omega \ll k$ limit, one can show that the spatial current $\mathbf{j}$ given by the right-hand side of Eq. (1.4) becomes, in the transverse sector,

$$\mathbf{j}_\mathrm{T} \simeq \frac{\pi m^2}{4k} \mathbf{E}_\mathrm{T} + (\text{higher order in } \mathbf{A}). \tag{1.5}$$

Fixing $A_0 = 0$ gauge, and working in Fourier space, Ampere's Law then becomes

$$(-\omega^2+k^2)\mathbf{A}_{\rm T}\simeq\frac{\pi m^2}{4k}\,i\omega\mathbf{A}_{\rm T}+(\text{higher order in }\mathbf{A}). \tag{1.6}$$

The coefficient of $\mathbf{A}_{\rm T}$ on the right-hand side is simply the $\omega\ll k$ limit of the transverse hard thermal loop self-energy [6]. For $\omega\ll k$, Eq. (1.6) becomes

$$k^2A_{\rm T}\sim\frac{m^2}{k}\,i\omega\mathbf{A}_{\rm T} \tag{1.7}$$

in orders of magnitude, if interactions are ignored. The characteristic frequency is then of order

$$|\omega|\sim\frac{k^3}{m^2}\quad\text{(ignoring interactions)}, \tag{1.8}$$

and we can now verify that this frequency indeed satisfies the assumed relationship $\omega\ll k$ when $k\ll m$. For this reason, in discussing effective theories for $k\ll m$, it is relevant and useful to also specialize to $\omega\ll k$. Interactions modify the estimate (1.8) when $k\ll\gamma$ [1,7], where $\gamma\sim g^2T\ln(1/g)$ is the inverse mean free time between color randomizing collisions, but the result that the characteristic frequency scale ω is small compared to k is unaffected.

The theory (1.4) represents an effective theory for momentum scales small compared to T. Bödeker has discussed what happens if one goes further and integrates out the physics down to some scale $\mu\ll m$. The hard particles which, microscopically, make up the color distributions W can have color-randomizing collisions by t-channel gluon exchange. Such collisions are dominated by momentum exchanges q in the range $g^2T\lesssim q\lesssim m$. Integrating out part of this momentum range generates an explicit collision term in the Boltzmann equation, replacing Eq. (1.3a) by

$$(D_t+\mathbf{v}\cdot\mathbf{D})W-\mathbf{v}\cdot\mathbf{E}=-\,\delta\hat{C}\,W+\xi, \tag{1.9a}$$

$$D_\nu F^{\mu\nu}=j^\mu=m^2\langle v^\mu W\rangle_{\mathbf{v}}. \tag{1.9b}$$

$\delta\hat{C}$ is a linearized collision operator. The magnitude of $\delta\hat{C}$ is logarithmically sensitive to the separation of the scales μ and m, and Bödeker has calculated $\delta\hat{C}$ at leading-order in that logarithm to be the local (in $\mathbf{x}$) operator defined by

$$\delta\hat{C}\,W(\mathbf{v})\equiv\langle\,\delta C(\mathbf{v},\mathbf{v}')\;W(\mathbf{v}')\rangle_{\mathbf{v}'}\,, \tag{1.10a}$$

$$\delta C(\mathbf{v},\mathbf{v}')\approx\gamma(\mu)\left[\delta^{S_2}(\mathbf{v}-\mathbf{v}')-\frac{4}{\pi}\,\frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}\right], \tag{1.10b}$$

$$\gamma(\mu)\approx C_{\rm A}\alpha T\ln\!\left(\frac{m}{\mu}\right). \tag{1.10c}$$

Here $\approx$ denotes equality at leading-log order, meaning that corrections are down by $[\ln(m/\mu)]^{-1}$, and δ^{S_2} is a δ-function on the unit sphere, normalized so that $\langle\,\delta^{S_2}(\mathbf{v}-\mathbf{v}')\rangle_{\mathbf{v}}=1$. To leading log order, $\gamma(\mu)$ is what's known as the hard thermal gluon damping rate [8] if one sets $\mu\sim g^2T$. This represents the inverse mean free path for color-randomizing collisions of the hard particles that, microscopically, make up the color distribution W.

The collision term in the Boltzmann equation damps the system towards equilibrium. In order to describe the physics of thermal fluctuations around equilibrium, one must also include a thermal noise term, which is the ξ shown in Eq. (1.9a). This equation is therefore an example of a Langevin equation. Bödeker derived the noise term, but one can also argue for it on general principles based on the fluctuation-dissipation theorem (for instance, along the lines of Ref. [7] or [9]). Bödeker found Gaussian white noise with correlation

$$\langle\langle\xi^a(\mathbf{v},\mathbf{x},t)\;\xi^b(\mathbf{v}',\mathbf{x}',t')\rangle\rangle=\frac{2T}{m^2}\,\delta C(\mathbf{v},\mathbf{v}')\;\delta^{ab}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')\times\delta(t-t'). \tag{1.11}$$

In writing formulas later on, it will be convenient to suppress indices and δ functions and write correlations like the above in the short-hand notation

$$\langle\langle\xi\xi\rangle\rangle=\frac{2T}{m^2}\,\delta\hat{C}. \tag{1.12}$$

The combination of Eqs. (1.9) and (1.11) make up Bödeker's effective theory for $k\ll m$. For Bödeker, this version of the theory was merely a stepping stone to deriving an even simpler and more infrared effective theory for $k\ll\gamma$, where W was eliminated. That theory is of the form

$$\mathbf{D}\times\mathbf{B}=\sigma\mathbf{E}+\boldsymbol{\zeta}, \tag{1.13a}$$

$$\langle\langle\zeta_i^a(\mathbf{x},t)\;\zeta_j^b(\mathbf{x}',t')\rangle\rangle=2\sigma T\;\delta_{ij}\;\delta^{ab}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\delta(t-t'). \tag{1.13b}$$

It has been used as the basis for numerical simulations to obtain the leading-log result for the hot electroweak baryon number violation rate [10].

Now return to the previous $k\ll m$ effective theory (1.9). The purpose of this paper is to present a cleaner, tidier version of this effective theory, more suitable for going beyond leading-log order in calculations. In particular, I shall (1) take the $\omega\ll k$ limit, discussed earlier, (2) show how to eliminate W from the result to obtain a single Langevin equation for $\mathbf{A}$, somewhat analogous to Eq. (1.4) but with damping and noise, and (3) show how to rewrite this Langevin equation as a path integral.

Part of the reason for wanting to take the $\omega\ll k$ limit is a pragmatic one. In field theory calculations, one tends to think of the philosophy of effective theories in the language of the Wilsonian renormalization group—''integrating out modes with $k\gtrsim\mu$.'' But a Wilson-style approach is generally impractical for perturbative calculations beyond lowest order, if one wants to set μ to be of order some characteristic scale of the problem. In practice, one usually *keeps* modes with $k\gg\mu$ and instead uses renormalization subtractions to achieve an equivalent result. Typically, dimensional regularization is used to regularize the ultraviolet. In an effective theory for scales $k\ll\mu$, it does not matter much what the physics is in the ultraviolet ($k \gg \mu$)—one adjusts the parameters of the effective theory to correct for the difference between the UV behavior of the effective theory and the UV behavior of the real theory. So, for instance, the bare $\delta\hat{C}$ in Eq. (1.9a) should be set to the *difference* between the collisions generated in the real theory due to gluon exchange with $q > \mu$, and those generated in the effective theory due to gluon exchange with $q > \mu$. The difficulty with Bödeker's $k \ll m$ effective theory as it stands is that, if one does not simply throw away the $k \gg \mu$ modes (which is difficult to do by hand in a gauge-invariant manner), then the equations (1.9) in fact reproduce all of the complicated $k \sim m$ behavior of the original hard thermal loop theory (1.4): plasmons, the Debye screening threshold, etc. Because of this, there is *no* difference between the $q > \mu$ contribution to $\delta\hat{C}$ in the two theories, and one should set the bare $\delta\hat{C}$ in Eq. (1.9a) to zero, returning right back to the original hard thermal loop description (1.9). For the leading-log calculations of Bödeker, none of this mattered—one could think of Wilsonian-style cutoffs at $k = \mu$, and all the associated difficulties are sub-leading order. To cleanly discuss effects beyond leading-log order, however, a more systematic approach to the $k \ll m$ effective theory is required, and it behooves us to reformulate the effective theory in a form where its UV behavior is as simple as possible and has no structure for $k \gg \mu$.

One of the other goals of this paper will be to reformulate the $k \ll m$ effective theory as a path integral. [As a warm-up, I will also review how to do the same for the simpler $k \ll \gamma$ effective theory of Eq. (1.13).] One reason this is useful is that path integrals provide, for many people, a more familiar starting point for calculations than do Langevin equations. Another reason is that one can fix gauges for perturbative calculations by the usual Faddeev-Popov procedure. The theory (1.13), for instance, was derived by Bödeker specifically in $A_0 = 0$ gauge. By converting the $A_0 = 0$ gauge result into a path integral and then generalizing the result to a gauge-invariant form, it will be easy to see how to correctly account for other, non-ghost-free gauge fixings, such as Coulomb gauge. Such gauges can be very convenient for calculations.

The advantages of the formalism discussed in this paper are put into use by me and Yaffe in Refs. [11,12], where we compute the next-to-leading-log corrections to Bödeker's far-infrared effective theory (1.13), and use it to analyze next-to-leading-logarithm corrections to the color conductivity and the hot electroweak baryon number violation rate.

Before continuing, I should be explicit about one technical point. Most of the various effective theories discussed in this paper are not ultraviolet finite and require regularization and renormalization [the one exception being Bödeker's final effective theory (1.13)]. I shall implicitly assume in this paper that divergences associated with small spatial scales have somehow been appropriately regulated. For instance, such divergences are regulated in Ref. [12] using dimensional regularization. I will, however, later focus explicitly on regularization issues associated with small time scales, which correspond to well-known ambiguities with certain types of Langevin equations and which may not be familiar to many readers.

II. PREVIEW OF RESULTS

I will recap Bödeker's original $k \ll m$ effective theory (1.3), now splitting Maxwell's equations into Gauss' law and Ampere's law:

$$(D_t + \mathbf{v}\cdot\mathbf{D})W - \mathbf{v}\cdot\mathbf{E} = -\delta\hat{C}\, W + \xi, \tag{2.1a}$$

$$\mathbf{D}\cdot\mathbf{E} = m^2 \langle W \rangle, \tag{2.1b}$$

$$-D_t \mathbf{E} + \mathbf{D}\times\mathbf{B} = m^2 \langle \mathbf{v} W \rangle. \tag{2.1c}$$

My result for appropriate equations in the $\omega \ll k$ limit, discussed in Sec. III, will be

$$\mathbf{v}\cdot\mathbf{D} W - \mathbf{v}\cdot\mathbf{E} = -\delta\hat{C}\, W + \xi, \tag{2.2a}$$

$$0 = m^2 \langle W \rangle, \tag{2.2b}$$

$$\mathbf{D}\times\mathbf{B} = m^2 \langle \mathbf{v} W \rangle. \tag{2.2c}$$

In Sec. IV, I discuss the form of Gauss' law (2.2b) and Ampere's law (2.2c) if the Boltzmann equation (2.2a) is used to eliminate W. In Sec. V, I then go on to show how Gauss' law and Ampere's law, together with the noise correlation (1.11), can be combined into a simple form analogous to Eq. (1.13),

$$\mathbf{D}\times\mathbf{B} = \overline{\sigma}(\mathbf{D})\, \mathbf{E} + \boldsymbol{\zeta}, \tag{2.3a}$$

$$\langle\langle \boldsymbol{\zeta}\boldsymbol{\zeta} \rangle\rangle = 2T\, \overline{\sigma}(\mathbf{D}), \tag{2.3b}$$

where the operator $\overline{\sigma}(\mathbf{D})$ will be defined later. This is an example of a Langevin equation with ''multiplicative noise,'' which simply means that the noise amplitude (2.3b) depends on the dynamical variable $\mathbf{A}$. Such equations are notorious for being ambiguous and sensitive to the details of ultraviolet regularization. In Sec. VI, I will address these issues, and show how to formulate the theory as a gauge-invariant path integral. The path integral has the form

$$Z = \int [\mathcal{D}A_0(\mathbf{x},t)][\mathcal{D}\mathbf{A}(\mathbf{x},t)] \exp\left(-\int dt\, d^3x\, L \right), \tag{2.4}$$

$$L = \frac{1}{4T} [-\overline{\sigma}(\mathbf{D})\, \mathbf{E} + \mathbf{D}\times\mathbf{B}]^{\mathrm{T}} \overline{\sigma}(\mathbf{D})^{-1} \times [-\overline{\sigma}(\mathbf{D})\, \mathbf{E} + \mathbf{D}\times\mathbf{B}] + L_1[\mathbf{A}]. \tag{2.5}$$

Very roughly speaking, the Gaussian integral in $-\overline{\sigma}(\mathbf{D})\, \mathbf{E} + \mathbf{D}\times\mathbf{B}$ implements a Gaussian probability distribution for $-\overline{\sigma}(\mathbf{D})\, \mathbf{E} + \mathbf{D}\times\mathbf{B}$, and so implements Eq. (2.3). The term $L_1[\mathbf{A}]$ is a complicated factor related to a Jacobian and to resolving the aforementioned ambiguities, and it will be discussed later.

III. THE $\omega \ll k$ LIMIT OF THE W EQUATIONS

The $\omega \ll k$ limit of the Boltzmann equation (2.1a) is easy to understand: we can ignore the $D_t W$ term compared to the

$\mathbf{v}\cdot\mathbf{D}W$ term. The resulting equation (2.2a) is no longer an evolution equation for W; instead, W is determined solely by the instantaneous values of $\mathbf{E}$ and ξ. Formally,

$$W=\hat{G}(\mathbf{v}\cdot\mathbf{E}+\xi) \tag{3.1a}$$

with

$$\hat{G}\equiv(\mathbf{v}\cdot\mathbf{D}+\delta\hat{C})^{-1}. \tag{3.1b}$$

Let us now analyze Gauss' law (2.1b) using this small ω approximation to W:

$$\mathbf{D}\cdot\mathbf{E}\simeq m^2\langle\hat{G}(\mathbf{v}\cdot\mathbf{E}+\xi)\rangle. \tag{3.2}$$

Again, the notation $\langle\cdots\rangle$ indicates averaging over $\mathbf{v}$-space, but one must carefully keep in mind that $\delta\hat{C}$ and $\hat{G}$ are operators in $\mathbf{v}$-space. This notation is that of Ref. [7], and the reader may find a thorough discussion of it in the introduction of Ref. [12]. It is now useful to split $\mathbf{E}$ into longitudinal and transverse pieces $\mathbf{E}_\mathrm{L}$ and $\mathbf{E}_\mathrm{T}$ [13], defined by the longitudinal and transverse projection operators

$$P_\mathrm{L}{}^{ij}=D^iD^{-2}D^j, \tag{3.3a}$$

$$P_\mathrm{T}{}^{ij}=\delta^{ij}-P_\mathrm{L}{}^{ij}, \tag{3.3b}$$

where i and j run over spatial indices and D^{-2} means $(\mathbf{D}\cdot\mathbf{D})^{-1}$. The order of magnitude of the left-hand side of Eq. (3.2) is then $O(kE_\mathrm{L})$. The right-hand side of Eq. (3.2) has, among other things, a term $m^2\langle\hat{G}\mathbf{v}\cdot\mathbf{E}_\mathrm{L}\rangle$ involving $\mathbf{E}_\mathrm{L}$. Using the projection operator (3.3a) and a frequently useful trick [12], this term can be rewritten as

$$\begin{aligned} m^2\langle\hat{G}\mathbf{v}\cdot\mathbf{E}_\mathrm{L}\rangle&=m^2\langle\hat{G}\mathbf{v}\cdot\mathbf{D}\rangle D^{-2}\mathbf{D}\cdot\mathbf{E}\\ &=m^2\langle\hat{G}(\mathbf{v}\cdot\mathbf{D}+\delta\hat{C})\rangle D^{-2}\mathbf{D}\cdot\mathbf{E}=m^2D^{-2}\mathbf{D}\cdot\mathbf{E}. \end{aligned} \tag{3.4}$$

The middle equality follows because $\delta\hat{C}$ has the property of annihilating functions that do not depend of $\mathbf{v}$, and so, as a general rule, $\langle\cdots\delta\hat{C}\rangle=0$ and $\langle\delta\hat{C}\cdots\rangle=0$. (See Refs. [1,13,12] for discussions of this.) From Eq. (3.4), we see that the $m^2\langle\hat{G}\mathbf{v}\cdot\mathbf{E}_\mathrm{L}\rangle$ term is $O(m^2k^{-2}\mathbf{D}\cdot\mathbf{E})$. That's bigger than the $\mathbf{D}\cdot\mathbf{E}$ term on the left-hand side of Eq. (3.2) by a factor of m^2/k^2, and m^2/k^2 is large for the modes whose physics I wish to correctly describe $(k\ll m)$. So it is permissible, when implementing the constraints of Gauss' law, to ignore the contribution of the $\mathbf{D}\cdot\mathbf{E}$ on the left-hand side, leaving

$$0\simeq m^2\langle\hat{G}(\mathbf{v}\cdot\mathbf{E}+\xi)\rangle. \tag{3.5}$$

Rewriting back in terms of W, this is the $\omega\ll k$ equation (2.2b) presented earlier.

Finally, consider Ampere's law (2.1c). For the moment, think about it in $A_0=0$ gauge, where it becomes

$$\partial_t^2\mathbf{A}+\mathbf{D}\times\mathbf{D}\times\mathbf{A}=m^2\langle\mathbf{v}W\rangle. \tag{3.6}$$

The first term is $O(\omega^2A)$ and the second $O(k^2A)$. This suggests that one may drop the first term in comparison to the second—at least in the transverse sector. ($\mathbf{D}\times\mathbf{B}=\mathbf{D}\times\mathbf{D}\times\mathbf{A}$ is purely transverse.) The result is the equation (2.2c) presented earlier. For this equation to be consistent, it had better be that the right-hand side is purely transverse as well (in the $\omega\ll k$ limit). Indeed,

$$\mathbf{D}\cdot\langle\mathbf{v}W\rangle=\langle\mathbf{v}\cdot\mathbf{D}W\rangle=\langle\mathbf{v}\cdot\mathbf{E}+\xi\rangle=\langle\xi\rangle, \tag{3.7}$$

where I have used the $\omega\ll k$ Boltzmann equation (2.2a). The $\mathbf{v}$-average $\langle\xi\rangle_\mathbf{v}$ of the noise ξ vanishes for the following reason [1]. Since ξ is Gaussian noise, so is $\langle\xi\rangle_\mathbf{v}$. But

$$\langle\langle\ \langle\xi\rangle\langle\xi\rangle\ \rangle\rangle=\langle\langle\ \langle\xi(\mathbf{v})\,\xi(\mathbf{v}')\rangle\ \rangle\rangle_{\mathbf{v},\mathbf{v}'}\propto\langle\delta\hat{C}(\mathbf{v},\mathbf{v}')\rangle_{\mathbf{v},\mathbf{v}'}=0, \tag{3.8}$$

so $\langle\xi\rangle$ is simply zero. Then Eq. (3.7) implies that $\langle\mathbf{v}W\rangle$ is indeed purely transverse in the $\omega\ll k$ effective theory.

IV. TWO EQUATIONS FOR A

There is a conceptual trap lurking in the $\omega\ll k$ equations (2.2) that is easy to fall into. Equation (2.2b) appears to say that $j^0=m^2\langle W\rangle$ vanishes in the $\omega\rightarrow0$ limit. And so, by Gauss' law, that $\mathbf{D}\cdot\mathbf{E}=0$. And one might take that to mean that longitudinal electric fields $\mathbf{E}_\mathrm{L}$ are negligible compared to transverse fields $\mathbf{E}_\mathrm{T}$ in the $\omega\ll k$ limit. This is incorrect.[1] Equation (2.2b) merely reflects the fact that the $\mathbf{D}\cdot\mathbf{E}$ term in Eq. (3.2) is negligible compared to the individual terms on the right-hand side of that equation—there is no presumption about how small $\mathbf{E}_\mathrm{L}$ is relative to $\mathbf{E}_\mathrm{T}$. It is perhaps less confusing to eliminate W altogether, and replace Eqs. (2.2) by the two equations

$$0=m^2\langle\hat{G}(\mathbf{v}\cdot\mathbf{E}+\xi)\rangle, \tag{4.1a}$$

$$\mathbf{D}\times\mathbf{B}=m^2\langle\mathbf{v}\hat{G}(\mathbf{v}\cdot\mathbf{E}+\xi)\rangle. \tag{4.1b}$$

The form (2.2) in terms of W has the advantage of having a more direct correspondence with the form of the original equations (1.9). I will want to refer to the W-eliminated form (4.1) in the next section, however, and so it is useful to simplify the noise terms in these equations. In particular, the terms $m^2\langle\hat{G}\xi\rangle$ and $m^2\langle\mathbf{v}\hat{G}\xi\rangle$ are proportional to Gaussian noise ξ and so are themselves Gaussian noise, and Gaussian noise can be completely specified just by specifying its correlator. So, rewrite the two equations (4.2) as

[1]Consider, for example, the case of $k\ll\gamma$, so that Eq. (1.13) gives an effective description of the physics, but $k\gg g^2T$, so that the physics is still perturbative. And consider, for example, the frequency scale $\omega\sim k^2/\sigma$. Then Eq. (1.13) gives the order of magnitude relation $\sigma\mathbf{E}\sim\boldsymbol{\zeta}$, which means that all polarizations of $\mathbf{E}$ are the same order of magnitude. See Ref. [13] for a detailed discussion of why the effective theory (1.13) applies to the longitudinal as well as transverse sector.

$$0=m^2\langle \hat{G}\mathbf{v}\rangle\cdot\mathbf{E}+\eta, \tag{4.2a}$$

$$\mathbf{D}\times\mathbf{B}=m^2\langle \mathbf{v}\hat{G}\mathbf{v}\rangle\cdot\mathbf{E}+\boldsymbol{\zeta}_{\rm T}, \tag{4.2b}$$

where

$$\langle\langle \eta\eta\rangle\rangle=m^4\langle \hat{G}\langle\langle \xi\xi\rangle\rangle\hat{G}^\top\rangle=2Tm^2\langle \hat{G}\ \delta\hat{C}\ \hat{G}^\top\rangle, \tag{4.3a}$$

$$\langle\langle \boldsymbol{\zeta}_{\rm T}\boldsymbol{\zeta}_{\rm T}\rangle\rangle=m^4\langle \mathbf{v}\hat{G}\langle\langle \xi\xi\rangle\rangle\hat{G}^\top\mathbf{v}\rangle=2Tm^2\langle \mathbf{v}\hat{G}\ \delta\hat{C}\ \hat{G}^\top\mathbf{v}\rangle. \tag{4.3b}$$

The right-hand sides implicitly have factors of $\delta(t-t')$, which I have suppressed. The transpose on $\hat{G}$ indicates transposition in **x**-space, color space, and **v**-space. D_i is the adjoint representation covariant derivative and satisfies $D_i^\top=-D_i$. The linearized collision operator $\delta\hat{C}$ is symmetric in **v**-space since, by rotation invariance, it can only depend[2] on $\mathbf{v}\cdot\mathbf{v}'$. [See Eq. (1.10b), for example, for the explicit version at leading log order.] So

$$\hat{G}^\top=(-\mathbf{v}\cdot\mathbf{D}+\delta\hat{C})^{-1}. \tag{4.4}$$

Now note that

$$\hat{G}\ \delta\hat{C}\ \hat{G}^\top=\tfrac{1}{2}\hat{G}[(\hat{G}^\top)^{-1}+(\hat{G})^{-1}]\hat{G}^\top=\tfrac{1}{2}(\hat{G}+\hat{G}^\top), \tag{4.5}$$

so

$$\langle\langle \eta\eta\rangle\rangle=Tm^2\langle \hat{G}+\hat{G}^\top\rangle, \tag{4.6}$$

$$\langle\langle \boldsymbol{\zeta}_{\rm T}\boldsymbol{\zeta}_{\rm T}\rangle\rangle=Tm^2\langle \mathbf{v}(\hat{G}+\hat{G}^\top)\mathbf{v}\rangle. \tag{4.7}$$

Taking $\mathbf{v}\to-\mathbf{v}$ in the **v** average shows that the $\hat{G}$ and $\hat{G}^\top$ terms give the same result, so that

$$\langle\langle \eta\eta\rangle\rangle=2Tm^2\langle \hat{G}\rangle, \tag{4.8a}$$

$$\langle\langle \boldsymbol{\zeta}_{\rm T}\boldsymbol{\zeta}_{\rm T}\rangle\rangle=2Tm^2\langle \mathbf{v}\hat{G}\mathbf{v}\rangle. \tag{4.8b}$$

Finally, we can verify that η and $\boldsymbol{\zeta}_{\rm T}$ are independent:

$$\langle\langle \eta\boldsymbol{\zeta}_{\rm T}\rangle\rangle=Tm^2\langle(\hat{G}+\hat{G}^\top)\mathbf{v}\rangle=0, \tag{4.9}$$

where the last equality follows by $\mathbf{v}\to-\mathbf{v}$ in the term involving $\hat{G}^\top$. The noise correlations (4.8), combined with the two equations (4.2) for **A**, give a complete description of the $\omega \ll k \ll m$ effective theory.

V. ONE EQUATION FOR A

It is possible to eliminate W and write a single equation for A that embodies all of Eq. (2.2) or Eq. (4.2). Define the projection operator $\hat{P}_0$ to be an operator in **v**-space that projects out functions that are independent of **v**; that is,

$$\hat{P}_0 f(\mathbf{v})=\langle f(\mathbf{v})\rangle. \tag{5.1}$$

The trick is to take the $1-\hat{P}_0$ projection of Eq. (2.2a), and to note that $\hat{P}_0$ vanishes on $\mathbf{v}\cdot\mathbf{E}$ and on ξ (because $\langle\xi\rangle=0$, as explained earlier). So

$$(1-\hat{P}_0)(\mathbf{v}\cdot\mathbf{D}+\delta\hat{C})W=\mathbf{v}\cdot\mathbf{E}+\xi. \tag{5.2}$$

Now note that Eq. (2.2b) tells us that $\hat{P}_0 W=0$, and so

$$(1-\hat{P}_0)(\mathbf{v}\cdot\mathbf{D}+\delta\hat{C})(1-\hat{P}_0)W=\mathbf{v}\cdot\mathbf{E}+\xi. \tag{5.3}$$

Then we can solve for W as

$$W=\hat{G}_1(\mathbf{v}\cdot\mathbf{E}+\xi), \tag{5.4a}$$

$$\hat{G}_1\equiv[(1-\hat{P}_0)(\mathbf{v}\cdot\mathbf{D}+\delta\hat{C})(1-\hat{P}_0)]^{-1}, \tag{5.4b}$$

where the inverse is understood to be taken in the space projected by $1-\hat{P}_0$. An alternative way to obtain the same inverse is

$$\hat{G}_1=\lim_{\Lambda\to\infty}(\mathbf{v}\cdot\mathbf{D}+\delta\hat{C}+\Lambda\hat{P}_0)^{-1}. \tag{5.4c}$$

Equation (5.4) appears different from the solution (3.1) used earlier for W. Indeed it *is* different for arbitrary **E**, but it produces the same W when **E** is such that the $\omega \ll k$ Gauss' law (2.2b) is satisfied. The advantage of the present form is that it may be used to derive a single equation containing all of the dynamics of the three equations (2.2). To proceed, use Eq. (5.4) in Ampere's law (2.2c) to get Eq. (2.3a),

$$\mathbf{D}\times\mathbf{B}=\bar{\sigma}(\mathbf{D})\ \mathbf{E}+\boldsymbol{\zeta}, \tag{5.5}$$

where $\bar{\sigma}(\mathbf{D})$ is a matrix in vector-index space,

$$\bar{\sigma}_{ij}(\mathbf{D})\equiv m^2\langle v_i\hat{G}_1 v_j\rangle=\lim_{\Lambda\to\infty}m^2\langle v_i(\mathbf{v}\cdot\mathbf{D}+\delta\hat{C}+\Lambda\hat{P}_0)^{-1}v_j\rangle, \tag{5.6}$$

and $\boldsymbol{\zeta}$ is Gaussian noise given by

$$\boldsymbol{\zeta}\equiv m^2\langle \mathbf{v}\hat{G}_1\xi\rangle=\lim_{\Lambda\to\infty}m^2\langle \mathbf{v}(\mathbf{v}\cdot\mathbf{D}+\delta\hat{C}+\Lambda\hat{P}_0)^{-1}\xi\rangle. \tag{5.7}$$

I will show in a moment that Eq. (5.5) subsumes the three equations (2.2), but first I will derive the correlation of the Gaussian noise $\boldsymbol{\zeta}$. Based on the analogy of Eq. (5.5) with the far-infrared effective theory (1.13), one might expect that the correlation is Eq. (2.3b). To verify it, start from the definition (5.7) of $\boldsymbol{\zeta}$, which gives

[2]This argument assumes that collisions do not depend on spin, or that spin has been averaged over. The $q\lesssim gT$ collisions that are of interest to this problem are indeed insensitive to spin at leading order in coupling.

$$\langle\langle \zeta_i \zeta_j \rangle\rangle = m^4 \langle v_i \hat{G}_1 \langle\langle \xi\xi \rangle\rangle \hat{G}_1^\top v_j \rangle = 2Tm^2 \langle v_i \hat{G}_1 \, \delta\hat{C} \, \hat{G}_1^\top v_j \rangle. \tag{5.8}$$

By arguments that parallel those used to derive the noise correlation (4.8b) of $\boldsymbol{\zeta}_{\rm T}$ from the analogous starting point (4.3b) in the previous section, one obtains

$$\langle\langle \zeta_i \zeta_j \rangle\rangle = 2Tm^2 \langle v_i \hat{G}_1 v_j \rangle = 2T \, \overline{\sigma}(\mathbf{D}). \tag{5.9}$$

I will now show that the simple equations I have derived,

$$\mathbf{D}\times\mathbf{B} = \overline{\sigma}(\mathbf{D}) \, \mathbf{E} + \boldsymbol{\zeta}, \tag{5.10a}$$

$$\langle\langle \boldsymbol{\zeta}\boldsymbol{\zeta} \rangle\rangle = 2T \, \overline{\sigma}(\mathbf{D}), \tag{5.10b}$$

provide a complete description of the $\omega \ll k \ll m$ effective theory originally described by Eq. (2.2) and the correlation (1.11). Specifically, I will show how to recover the two individual equations (4.2) of the last section for Gauss' law and Ampere's law, which were the result of trivially eliminating W from Eq. (2.2). It is convenient to first establish a relation between the $(1-\hat{P}_0)$ projected W propagator $\hat{G}_1$ of Eqs. (5.4b) and (5.4c) and the original unprojected propagator of Eq. (3.1b). The relation is

$$\hat{G}_1 = \hat{G} - \hat{G}\hat{P}_0 \langle \hat{G} \rangle^{-1} \hat{G}, \tag{5.11}$$

which can also be thought of as the rule

$$\langle \cdots (\hat{G}_1 - \hat{G}) \cdots \rangle = -\langle \cdots \hat{G} \rangle \, \langle \hat{G} \rangle^{-1} \, \langle \hat{G} \cdots \rangle, \tag{5.12}$$

e.g.

$$\overline{\sigma}(\mathbf{D}) = m^2 \langle \mathbf{v}\hat{G}_1 \mathbf{v} \rangle = m^2 [\langle \mathbf{v}\hat{G}\mathbf{v} \rangle - \langle \mathbf{v}\hat{G} \rangle \langle \hat{G} \rangle^{-1} \langle \hat{G}\mathbf{v} \rangle]. \tag{5.13}$$

It is easy to verify Eq. (5.11) by first checking that it lives in the space projected by $1-\hat{P}_0$,

$$\begin{aligned} \hat{P}_0[\hat{G} - \hat{G}\hat{P}_0 \langle G \rangle^{-1} \hat{G}] &= \hat{P}_0 G - \hat{P}_0 \langle G \rangle \langle G \rangle^{-1} \hat{G} \\ &= \hat{P}_0 G - \hat{P}_0 G = 0 \end{aligned} \tag{5.14}$$

(and similarly $[\hat{G} + \hat{G}\hat{P}_0 \langle G \rangle^{-1} \hat{G}]\hat{P}_0 = 0$), and then checking explicitly that it is the desired inverse:

$$\begin{aligned} &[\hat{G} - \hat{G}\hat{P}_0 \langle G \rangle^{-1} \hat{G}]\hat{G}_1^{-1} \\ &\quad = (1-\hat{P}_0)[\hat{G} - \hat{G}\hat{P}_0 \langle G \rangle^{-1} \hat{G}]\hat{G}^{-1}(1-\hat{P}_0) \\ &\quad = (1-\hat{P}_0)[1 - \hat{G}\hat{P}_0 \langle G \rangle^{-1}](1-\hat{P}_0) \\ &\quad = (1-\hat{P}_0). \end{aligned} \tag{5.15}$$

Now I will show that the gauge field Langevin equation (5.10a) implies Gauss' law (4.2a) by dotting $\langle G \rangle \mathbf{D}$ into both sides of Eq. (5.10a):

$$0 = \langle G \rangle [\mathbf{D} \, \overline{\sigma}(\mathbf{D}) \, \mathbf{E} + \mathbf{D}\cdot\boldsymbol{\zeta}]. \tag{5.16}$$

Using Eq. (5.13),

$$\langle G \rangle \mathbf{D} \, \overline{\sigma}(\mathbf{D}) \, \mathbf{E} = m^2 \langle G \rangle [\langle \mathbf{v}\cdot\mathbf{D}\hat{G}\mathbf{v} \rangle - \langle \mathbf{v}\cdot\mathbf{D}\hat{G} \rangle \langle \hat{G} \rangle^{-1} \langle \hat{G}\mathbf{v} \rangle]\mathbf{E}. \tag{5.17}$$

We can simplify using the trick discussed earlier,

$$\langle \mathbf{v}\cdot\mathbf{D}\hat{G} \cdots \rangle = \langle (\mathbf{v}\cdot\mathbf{D} + \delta\hat{C})\hat{G} \cdots \rangle = \langle \cdots \rangle, \tag{5.18}$$

so that

$$\langle \mathbf{v}\cdot\mathbf{D}\hat{G}\mathbf{v} \rangle = \langle \mathbf{v} \rangle = 0, \tag{5.19a}$$

$$\langle \mathbf{v}\cdot\mathbf{D}\hat{G} \rangle = 1. \tag{5.19b}$$

Equation (5.17) for the $\overline{\sigma}$ term then becomes

$$\langle G \rangle \mathbf{D} \, \overline{\sigma}(\mathbf{D}) \, \mathbf{E} = -m^2 \langle \hat{G}\mathbf{v} \rangle \cdot \mathbf{E}. \tag{5.20}$$

So Eq. (5.16) becomes

$$0 = m^2 \langle \hat{G}\mathbf{v} \rangle \cdot \mathbf{E} - \langle G \rangle \mathbf{D}\cdot\boldsymbol{\zeta}. \tag{5.21}$$

Compare to Gauss' law (4.2a). The last term is Gaussian noise, and all that matters for the purpose of reproducing Gauss' law (4.2a) is to check that $\eta' \equiv -\langle G \rangle \mathbf{D}\cdot\boldsymbol{\zeta}$ has the same noise correlation (4.8a) as η of the last section. First put together Eqs. (5.11) and (5.18) to get

$$\langle \mathbf{v}\cdot\mathbf{D}\hat{G}_1 \cdots \rangle = \langle [1 - \langle \hat{G} \rangle^{-1} \hat{G}] \cdots \rangle. \tag{5.22}$$

Then, using the $\boldsymbol{\zeta}$ correlation (5.10b),

$$\begin{aligned} \langle\langle \eta' \, \eta' \rangle\rangle &= \langle \hat{G} \rangle \mathbf{D}\cdot\langle\langle \boldsymbol{\zeta}\boldsymbol{\zeta} \rangle\rangle \cdot \mathbf{D}^\top \langle \hat{G}^\top \rangle \\ &= -2Tm^2 \langle \hat{G} \rangle \langle \mathbf{v}\cdot\mathbf{D}\hat{G}_1 \mathbf{v}\cdot\mathbf{D} \rangle \langle \hat{G} \rangle \\ &= 2Tm^2 \langle \hat{G}\mathbf{v}\cdot\mathbf{D} \rangle \langle \hat{G} \rangle \\ &= 2Tm^2 \langle \hat{G} \rangle = \langle\langle \eta\eta \rangle\rangle. \end{aligned} \tag{5.23}$$

Alternatively, one could go back to the expression (5.7) for $\boldsymbol{\zeta}$ in terms of ξ, and show directly that $\eta' \equiv -\langle G \rangle \mathbf{D}\cdot\boldsymbol{\zeta}$ is the same as the $\eta \equiv m^2 \langle \hat{G}\xi \rangle$.

Having obtained Gauss' law, we can now check that the single equation (5.10a) also enforces Ampere's law (4.2b). Expand $\overline{\sigma}$ using Eq. (5.13), so that Eq. (5.10a) becomes

$$\begin{aligned} \mathbf{D}\times\mathbf{B} &= m^2 \langle \mathbf{v}\hat{G}\mathbf{v} \rangle \cdot \mathbf{E} - m^2 \langle \mathbf{v}\hat{G} \rangle \langle \hat{G} \rangle^{-1} \langle \hat{G}\mathbf{v} \rangle \cdot \mathbf{E} + \boldsymbol{\zeta} \\ &= m^2 \langle \mathbf{v}\hat{G}\mathbf{v} \rangle \cdot \mathbf{E} + \boldsymbol{\zeta}'_{\rm T}, \end{aligned} \tag{5.24}$$

where the last equality uses Gauss' law (5.21) and defines

$$\boldsymbol{\zeta}'_{\rm T} \equiv \boldsymbol{\zeta} - \langle \mathbf{v}G \rangle \mathbf{D}\cdot\boldsymbol{\zeta}. \tag{5.25}$$

One may verify that the noise $\boldsymbol{\zeta}'_{\rm T}$ is equivalent to the noise $\boldsymbol{\zeta}_{\rm T}$ (4.8b) of the previous section. It is worth mentioning that $\boldsymbol{\zeta}'_{\rm T}$ is transverse $(\mathbf{D}\cdot\boldsymbol{\zeta}'_{\rm T}=0)$, but it is *not* simply the transverse projection $P_{\rm T}\boldsymbol{\zeta}$ of $\boldsymbol{\zeta}$.

VI. PATH INTEGRALS AND AMBIGUITIES

A. A warmup: the $k \ll \gamma$ theory

1. The path integral in $A_0=0$ gauge

To warm up to talking about path integral formulations for the $\omega \ll k \ll m$ Langevin equation (2.3a), I will start by discussing path integrals for the simpler, more infrared effective theory described by Eq. (1.13) for $\omega \ll k \ll \gamma$. In $A_0=0$ gauge,

$$\sigma\dot{\mathbf{A}} = -\mathbf{D}\times\mathbf{B}+\boldsymbol{\zeta}, \tag{6.1a}$$

$$\langle\langle\boldsymbol{\zeta}\boldsymbol{\zeta}\rangle\rangle = 2\sigma T. \tag{6.1b}$$

In this gauge, the above Langevin equation has a nice physical interpretation, because it can be rewritten as

$$\sigma\dot{\mathbf{A}} = -\frac{\delta}{\delta\mathbf{A}}\mathcal{V}[\mathbf{A}]+\boldsymbol{\zeta}, \tag{6.2}$$

where

$$\mathcal{V}[\mathbf{A}] = \int d^3x\,\tfrac{1}{2}\mathbf{B}^a\cdot\mathbf{B}^a \tag{6.3}$$

is the magnetic energy. This means the Langevin equation is just an infinite degree of freedom version of the kinematics of a highly damped particle in a potential $V(\mathbf{q})$:

$$\sigma\dot{q}_i = -\frac{d}{dq_i}V(\mathbf{q})+\zeta_i, \tag{6.4a}$$

$$\langle\langle\zeta_i(t)\,\zeta_j(t')\rangle\rangle = 2\sigma T\delta_{ij}\,\delta(t-t'). \tag{6.4b}$$

It is well known how to rewrite such equations, and their field theory counterparts, as path integrals,[3] but I will briefly review the steps here. Keep to the notation (6.4) for the moment, and first consider an integral over the distribution for the Gaussian noise:

$$Z \equiv \int[\mathcal{D}\boldsymbol{\zeta}(t)]\exp\left[-\frac{1}{4\sigma T}\int dt|\boldsymbol{\zeta}(t)|^2\right]. \tag{6.5}$$

Now insert a factor of one in the form of the equation of motion (6.4a):

$$Z = \int[\mathcal{D}\boldsymbol{\zeta}(t)]\exp\left[-\frac{1}{4\sigma T}\int dt|\boldsymbol{\zeta}(t)|^2\right]\int[\mathcal{D}\mathbf{q}(t)]\times\delta[\sigma\dot{\mathbf{q}}+\boldsymbol{\nabla}_{\mathbf{q}}V(\mathbf{q})-\boldsymbol{\zeta}]J[\mathbf{q}], \tag{6.6}$$

where the δ function is functional, and the corresponding Jacobian is given by a functional determinant

$$J[\mathbf{q}] \equiv \det_{ij}\left(\frac{d}{dq_i}[\sigma\dot{q}_j+\nabla_{q_j}V(\mathbf{q})-\zeta]\right) = \det_{ij}\left(\sigma\delta_{ij}\frac{d}{dt}+\nabla_{q_i}\nabla_{q_j}V(\mathbf{q})\right). \tag{6.7}$$

Now use the δ-function to perform the noise integral in Eq. (6.6), to get

$$Z = \int[\mathcal{D}\mathbf{q}(t)]J[\mathbf{q}]\exp\left[-\frac{1}{4\sigma T}\int dt|\sigma\dot{\mathbf{q}}+\boldsymbol{\nabla}_{\mathbf{q}}V(\mathbf{q})|^2\right]. \tag{6.8}$$

One may simplify the Jacobian further, but the details depend on how one regularizes short times in the path integral. That is, there is sensitivity to what convention one uses for discretizing time in the path integral. [In contrast, the original Langevin equation (6.4) is insensitive to the details of short-time regularization.] If one makes the standard choice of a time-symmetric discretization scheme, where $\dot{\mathbf{q}}$ and $\mathbf{q}$ in the path integral are interpreted as

$$\dot{\mathbf{q}} = \frac{q(t_i)-q(t_{i-1})}{\Delta t},\qquad \mathbf{q} = \frac{q(t_i)+q(t_{i-1})}{2}, \tag{6.9}$$

then one may show that the Jacobian simplifies to [14,15]

$$J[\mathbf{q}] = \exp\left[-\frac{\theta(0)}{\sigma}\int dt\nabla^2_{\mathbf{q}}V(\mathbf{q})\right] \tag{6.10}$$

with the symmetric interpretation

$$\theta(0) = \tfrac{1}{2} \tag{6.11}$$

of the step function $\theta(t)$.

In field theory, $\mathbf{q}$ becomes $\mathbf{A}$, and $\mathbf{q}$ derivatives become functional derivatives. The path integral (6.8) becomes, in the case at hand,

$$Z = \int[\mathcal{D}\mathbf{A}]J[\mathbf{A}]\exp\left[-\frac{1}{4\sigma T}\int dt\,d^3x|\sigma\dot{\mathbf{A}}+\mathbf{D}\times\mathbf{B}|^2\right], \tag{6.12}$$

$$J[\mathbf{A}] = \exp\left[-\frac{\theta(0)}{\sigma}\int dt\,d^3x\frac{\delta}{\delta A^a_i(\mathbf{x})}(\mathbf{D}\times\mathbf{B})^a_i(\mathbf{x})\right] = \exp\left[-\frac{\theta(0)}{\sigma}\delta^{(3)}(0)\int dt\,d^3x\frac{d}{dA^a_i}(\mathbf{D}\times\mathbf{B})^a_i\right]. \tag{6.13}$$

It is easy enough to take the derivative, to get

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[3]For a review, see, for example, Chaps. 4 and 17 of Ref. [14]. In that reference, the determinant in Eq. (6.7) is implemented by ghost fields (which are unrelated to gauge fixing). That is, $J[\mathbf{q}] = \int[\mathcal{D}\overline{\mathbf{b}}][\mathcal{D}\mathbf{b}]\exp[-\int dt\bar{b}_i(\sigma\delta_{ij}\partial_t+\nabla_{q_i}\nabla_{q_j}V(\mathbf{q}))b_j]$ where I have labeled the ghosts $\overline{\mathbf{b}}$ and $\mathbf{b}$.

$$J[\mathbf{A}]=\exp[-\sigma^{-1}\delta^{(3)}(0)\,\mathrm{tr}\mathbf{D}^2], \tag{6.14}$$

but it is unnecessary if one uses dimensional regularization. In dimensional regularization, $\delta^{(d)}(0)$ vanishes (where I take $d=3-\epsilon$ to be the number of spatial dimensions), and so

$$J[\mathbf{A}]=1 \quad \text{(dimensional regularization).} \tag{6.15}$$

This is a feature of *any* Langevin field equation that is local in space.

2. The path integral in other gauges

Knowing the result in $A_0=0$ gauge, it is easy to guess the corresponding path integral *without* gauge-fixing:[4]

$$Z=\int[\mathcal{D}A_0][\mathcal{D}\mathbf{A}]J[\mathbf{A}]\times\exp\left[-\frac{1}{4\sigma T}\int dt\,d^3x\,|\sigma\mathbf{E}+\mathbf{D}\times\mathbf{B}|^2\right]. \tag{6.16}$$

This can be verified by now fixing $A_0=0$ gauge in the usual way, and obtaining Eq. (6.12). The advantage of the gauge-invariant form is that one can now alternatively fix other gauges in the usual way, by introducing Faddeev-Popov ghosts $\mathbf{c}$ and $\bar{\mathbf{c}}$. For example, to fix Coulomb gauge,

$$Z=\int[\mathcal{D}A_0][\mathcal{D}\mathbf{A}][\mathcal{D}\bar{\mathbf{c}}][\mathcal{D}\mathbf{c}]\,\delta(\boldsymbol{\nabla}\cdot\mathbf{A})J[\mathbf{A}]\times\exp\left(-\int dt\,d^3x\,L_{\mathrm{Coulomb}}\right), \tag{6.17}$$

$$L_{\mathrm{Coulomb}}=\frac{1}{4\sigma T}[|-\sigma\mathbf{E}+\mathbf{D}\times\mathbf{B}|^2+\bar{\mathbf{c}}\boldsymbol{\nabla}\cdot\mathbf{D}\mathbf{c}]. \tag{6.18}$$

B. The $k\ll m$ theory

There are two important differences between the $k\ll m$ Langevin equation (2.3) and the simpler $k\ll\gamma$ equation (1.13). The first is that the $k\ll m$ equation is non-local, which means that the Jacobian term in the path integral, analogous to Eq. (6.14), will not involve $\delta^{(3)}(0)$ and so will not trivially vanish in dimensional regularization. The second is that the amplitude of the damping and the noise in the $k\ll m$ equation depends on the state $\mathbf{A}$ of the system. As mentioned before, this means that the continuum Langevin equation does not have a well-defined meaning. The Langevin equation itself (and not just the path integral description) is sensitive to the ultraviolet and details of UV frequency regularization.

The fact that an effective theory is sensitive to details of ultraviolet regularization is not novel. Almost all effective field theories require ultraviolet regularization, and it was only the anomalous fact that the $k\ll\gamma$ effective theory (1.13) happens to be ultraviolet *finite*[5] that meant we did not need to regularize it. As usual, one should simply pick a regularization scheme and then fix the regularized parameters of the effective theory so that it reproduces the infrared physics of whatever more fundamental theory underlies it. In Ref. [17], I have discussed this matching problem for general systems of the form

$$\sigma_{ij}(\mathbf{q})\,\dot{q}_j=-\nabla_{q_i}V(\mathbf{q})+\zeta_i, \tag{6.19a}$$

$$\langle\langle\zeta_i(t)\,\zeta_j(t')\rangle\rangle=2T\,\sigma_{ij}(\mathbf{q})\,\delta(t-t'), \tag{6.19b}$$

in cases where it is known that the equilibrium distribution for $\mathbf{q}$ is

$$P_{\mathrm{eq}}(\mathbf{q})=e^{-V(\mathbf{q})/T} \tag{6.20}$$

in whatever approximation one is working in. This is useful in the present case because static equilibrium properties of hot gauge theories are much simpler to analyze than dynamical ones, and indeed the equilibrium distribution in $A_0=0$ gauge should be Eq. (6.20) with V the magnetic energy. In Ref. [17], I discuss how knowledge of the equilibrium distribution (6.20) forces the ambiguities inherent in the continuum Langevin equations (6.19) to be resolved in a particular way. I also showed that the corresponding path integral formulation is

$$Z=\int[\mathcal{D}\mathbf{q}(t)]\exp\left[-\int dt\,L(\dot{\mathbf{q}},\mathbf{q})\right], \tag{6.21}$$

$$L(\dot{\mathbf{q}},\mathbf{q})=\frac{1}{4T}(\sigma\dot{\mathbf{q}}+\boldsymbol{\nabla}_{\mathbf{q}}V)^{\top}\sigma^{-1}(\sigma\dot{\mathbf{q}}+\boldsymbol{\nabla}_{\mathbf{q}}V)+L_1(\mathbf{q}), \tag{6.22}$$

$$L_1(\mathbf{q})=-\frac{1}{2}\nabla_{q_i}[(\sigma^{-1})_{ij}\nabla_{q_j}V]+\frac{T}{4}\nabla_{q_i}\nabla_{q_j}(\sigma^{-1})_{ij}-\frac{1}{2}\delta(0)\,\mathrm{tr}\ln\sigma, \tag{6.23}$$

if the path integral is defined with symmetric time discretization (6.9). $\delta(0)$ above is short-hand for $\delta(t=0)=(\Delta t)^{-1}$. The first term in Eq. (6.22) is the obvious generalization of the exponent in Eq. (6.8) from scalar σ to matrix $\sigma_{ij}(\mathbf{q})$. The remaining $L_1(\mathbf{q})$ term represents the appropriate Jacobian (more accurately, $-\ln J$) and the terms necessary for the desired resolution of the ambiguities of the continuum Langevin equation (6.19). One may easily verify that specialization to the case $\sigma_{ij}(\mathbf{q})=\sigma\delta_{ij}$, with σ constant, reproduces the earlier result (6.8) [up to an irrelevant constant normalization].

In $A_0=0$ gauge, we can now obtain the path integral for the gauge theory case by replacing $\mathbf{q}$ by $\mathbf{A}$ and derivatives by functional derivatives. The resulting action density is

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[4]See the discussion surrounding Eq. (4.9) of Ref. [16].

[5]For a discussion in the present context, see Ref. [7].

$$L = \frac{1}{4T}[\overline{\sigma}(\mathbf{D})\,\dot{\mathbf{A}} + \mathbf{D}\times\mathbf{B}]^{\mathrm{T}}\overline{\sigma}(\mathbf{D})^{-1} \times[\overline{\sigma}(\mathbf{D})\,\dot{\mathbf{A}} + \mathbf{D}\times\mathbf{B}] + L_1[\mathbf{A}], \tag{6.24}$$

which, except for the $L_1[\mathbf{A}]$ term, is the natural generalization of the $k \ll \gamma$ action in Eq. (6.12). The $L_1[\mathbf{A}]$ term, however, is ugly as sin. So much so, that it is unilluminating to write it down, other than to refer back to the discrete version (6.23). I have been unable to find an attractive form for $L_1[\mathbf{A}]$. Fortunately, $L_1[\mathbf{A}]$ does not enter at all into certain important applications of this formalism, as I will discuss shortly.

A gauge-unfixed version of the action (6.22) can be found simply by finding a gauge-invariant action that becomes Eq. (6.22) when fixed to $A_0 = 0$ gauge. The result is

$$L = \frac{1}{4T}[-\overline{\sigma}(\mathbf{D})\,\mathbf{E} + \mathbf{D}\times\mathbf{B}]^{\mathrm{T}}\overline{\sigma}(\mathbf{D})^{-1} \times[-\overline{\sigma}(\mathbf{D})\,\mathbf{E} + \mathbf{D}\times\mathbf{B}] + L_1[\mathbf{A}]. \tag{6.25}$$

This action may then be used to fix whatever gauge is desired.

Note that $L_1[\mathbf{A}]$, though derived in $A_0 = 0$ gauge, is guage-invariant under general time-dependent gauge transformations. The derivation in $A_0 = 0$ gauge implied that $L_1[\mathbf{A}]$ is invariant under time-*independent* gauge transformations. Because $L_1[\mathbf{A}]$ does not involve any time derivatives (and because I did not introduce A_0 into this term), it is then automatically invariant under time-dependent transformations as well.

C. The nature of $L_1[\mathbf{A}]$

The coupling constant g is a convenient parameter for counting powers of the loop expansion. At high temperature, the parameter which controls the effectiveness of the loop expansion is not g^2 by itself, but it is at least proportional to an explicit factor of g^2. For analysis of static equilibrium quantities, for example, the loop expansion parameter is g^2T/k (once appropriate resummations have been implemented) for momenta $k \gtrsim g^2T$. The fact that physics is somehow treatable perturbatively for $k \gg g^2T$ (after integrating out degrees of freedom that decouple at various physical thresholds) is a reflection of the fact that the size of gauge field fluctuations is perturbatively small for such k. In Refs. [11,12], for example, Yaffe and I use the loop expansion of the $\omega \ll k \ll m$ theory at $k \sim \gamma$ to compute corrections to color conductivity and hot electroweak baryon number violation. The loop expansion is in that case an expansion in $g^2T/k \sim [\ln(1/g)]^{-1}$.

To understand at what order in the loop expansion interactions in the Lagrangian might contribute, it is therefore important to understand what explicit factors of g are associated with those interactions. Let us focus in particular on the (horrible) terms of $L_1[\mathbf{A}]$. First, note that $\mathbf{A}$ only appears in the combination $g\mathbf{A}$ in $\sigma(\mathbf{D})$, since $\mathbf{D} = \boldsymbol{\nabla} + g\mathbf{A}$. So expanding $\sigma(\mathbf{D})$ in powers of g gives

$$\sigma(\mathbf{D}) = \sigma(\boldsymbol{\nabla}) + O(gA) + O(g^2A^2) + \cdots, \tag{6.26}$$

and then similarly,

$$[\sigma(\mathbf{D})]^{-1} = [\sigma(\boldsymbol{\nabla})]^{-1} + O(gA) + O(g^2A^2) + \cdots, \tag{6.27}$$

where I am only keeping track of the explicit powers of g and $\mathbf{A}$ at each order. The terms in this expansion are *not* local in space.

One can now read off, for instance, that the gauge theory term corresponding to the $\nabla_{q_i}\nabla_{q_j}(\sigma^{-1})_{ij}$ term in Eq. (6.23) for L_1 must be schematically of the form

$$\frac{\delta^2}{\delta A^2}\sigma^{-1} = O(g^2) + O(g^3A) + O(g^4A^2) + \cdots. \tag{6.28}$$

The $O(g^2)$ term is independent of $\mathbf{A}$ and so can be discarded. There cannot actually be an $O(g^2A)$ term in the Lagrangian because there is no way for it to be a color singlet. So the leading piece of the $\delta^2\sigma^{-1}/\delta A^2$ term of $L_1[\mathbf{A}]$ is $O(g^4A^2)$. The explicit factor of g^4 implies that it will be suppressed by two powers of the loop expansion parameter compared to the $O(g^0A^2)$ terms in the action (6.25), which determine the $\mathbf{A}$ propagator. [And the $\mathbf{A}^3$ and so forth terms are similarly suppressed compared to non-L_1 $\mathbf{A}^3$ and so forth terms in Eq. (6.25).]

Similarly, the magnetic energy is

$$V = O(A^2) + O(gA^3) + O(g^2A^4). \tag{6.29}$$

The possible terms arising from the $\nabla_{\mathbf{q}}[\sigma^{-1}\nabla_{\mathbf{q}}V]$ term in Eq. (6.23) are then of the form

$$\frac{\delta}{\delta A}\left[\sigma^{-1}\frac{\delta}{\delta A}V\right] = O(1) + O(gA) + O(g^2A^2) + \cdots. \tag{6.30}$$

Again, $O(1)$ can be discarded, and $O(gA)$ cannot appear in the action, so the leading term in g must be $O(g^2A^2)$. This is suppressed by one power of the loop expansion parameter compared to the $O(g^0A^2)$ terms in the action (6.25). The analysis of the remaining term, $\ln\sigma$, is similar.

The conclusion is that the interactions among $\mathbf{A}$ generated by $L_1[\mathbf{A}]$ will all be suppressed by at least one power of the loop expansion parameter, compared to those appearing in the other terms of Eq. (6.25). In Ref. [12], Yaffe and I show that this suppression is enough to permit a next-to-leading-log order analysis of the color conductivity and the hot electroweak baryon number violation rate without requiring use of an explicit form for $L_1[\mathbf{A}]$.

A word of caution about the above analysis is required, however. The correspondence between explicit powers of g^2 and the expansion parameter g^2T/k only works if one has an effective theory that properly integrates out all of the physics above the scale you are interested in. For example, if one does perturbation theory in the original $k \ll T$ hard-thermal loop effective theory (1.4), the loop expansion will break down at $k \sim \gamma$: the loop expansion parameter will be $O(1)$

instead of order $g^2T/k \sim g^2T/\gamma \sim [\ln(1/g)]^{-1}$. One must instead make the loop expansion in the $k \ll m$ effective theory, which incorporates the effects of collisions into the bare propagators and vertices. As long as the correct effective theory is used, there *should* be no problem. However, for the sake of caution, it would be useful to have a much more explicit analysis of the suppression of the $L_1[\mathbf{A}]$ terms than I have been able to give.

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